Assignment 3

1. Find the linear polynomial interpolating the points (0.3, 4.5) and (0.7, 0.4).

$-10.25x + 7.575$, or if you are shifting to the zero, you get $-10.25(x - 0.3) + 4.5$

2. Find the quadratic polynomial interpolating the points (0, 5.2), (1, 2.6) and (3, 4.7).

$1.216666667x^2 - 3.816666667x + 5.2$

3. Find the quadratic polynomial interpolating the points $(-1, y_{-1})$, $(0, y_0)$ and $(1, y_1)$.

$\frac{1}{2}(y_1 - 2y_0 + y_{-1})x^2 + \frac{1}{2}(y_1 - y_{-1})x + y_0$

4. What is the condition number of the Vandermonde matrix for finding the interpolating quadratic polynomial interpolating the points $(15235, y_0)$, $(15240, y_1)$ and $(15245, y_2)$, and how does that contrast with the condition number of the matrix in Question 3?

From Matlab, the condition number is a few orders of magnitude larger:

```
>> cond( vander( [15235 15240 15245] ) )
     ans = 4.577266405745841e+15
>> cond( vander( [-1 0 1] ) )
     ans = 3.225504926677695
```

5. Up to this point, we have discussed interpolating polynomials. Suppose we had two points (0.3, 4.5) and (0.7, 0.4) but now we wanted to find a trigonometric function of the form $y = a\cos(x) + b\sin(x)$ that passes through these two points. What are the two linear equations that must be solved, and use Matlab to find the coefficients a and b and then test your solution by substituting in $x = 0.3$ and $x = 0.7$ into the resulting trigonometric function.

$a\cos(0.3) + b\sin(0.3) = 4.5$
$a\cos(0.7) + b\sin(0.7) = 0.4$

In Matlab:

```
>> a = [cos(0.3) sin(0.3); cos(0.7) sin(0.7)] \ [4.5 0.4]'
    a =
       7.140833411747852
      -7.856988011892086
>> a(1)*cos(0.3) + a(2)*sin(0.3)   % This should be 4.5
ans =  4.500000000000000
>> a(1)*cos(0.7) + a(2)*sin(0.7)   % This should be 0.4
ans =    4.000000000000004e-01
```

6. Write down the 1st-order Taylor series for sin(2 + *h*). Use this to approximate the value of sin(2.1), and approximate the specific value of ξ so that the formula is exact given that sin(2.1) = 0.8632093666488738 to sixteen significant digits. You can use your calculator and you will need the arc-sine function.

$$\sin\ 2\ \ + 0.1 \cdot \cos\ 2\ \ = 0.8676827431709675$$

and the error of this approximation is the exact minus the approximation:

$$0.8632093666488738 - 0.8676827431709675 = -0.0044733765220937.$$

The error is $\frac{1}{2} f^{\ 2}\ \ \xi\ h^2$ so we want to find the point such that $-\frac{1}{2}\sin\ \xi\ \ 0.1^2 = -0.0044733765220937$ so we must calculate $\xi = \sin^{-1}\ 2 \cdot 0.44733765220937\ \ =\ \ 1.107703824436590$, but this is not on the interval [2, 2.1], and so realize that sine is 2π periodic, so $2\pi n + 1.1077\cdots$ are all points that map to $2 \cdot 0.4473\cdots$, but also due to symmetry, so are all points of the form $2\pi n + \pi - 1.1077\cdots$. If we choose *n* = 0 in this second form, we get ξ = 2.033888829153203, which lies between 2 and 2.1.

7. Write down the 2nd-order Taylor series for e^{1+h}. Use this to approximate the value of $e^{0.9}$, and approximate the specific value of ξ so that the formula is exact given that $e^{0.9}$ = 2.459603111156950 to sixteen significant digits. You can use your calculator and you will need the natural logarithm function.

$$e^1 - 0.1 \cdot e^1 + \frac{1}{2} 0.1^2 \cdot e^1 = 2.460045054755436$$

and the error of this approximation is the exact minus the approximation:

$$2.459603111156950 - 2.460045054755436 = -0.000441943598486.$$

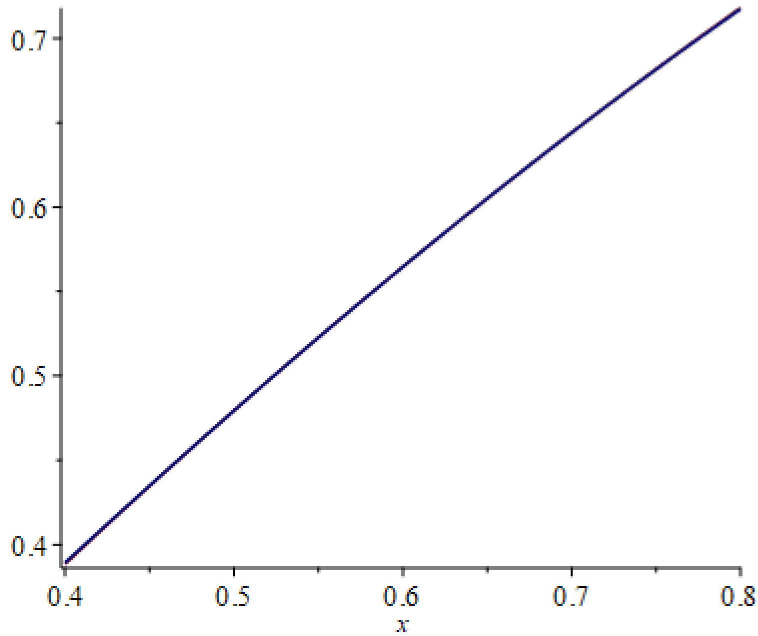
The error is $\frac{1}{6} f^{\ 3}\ \ \xi\ h^3$ so we want to find the point such that $\frac{1}{6} e^{\xi}\ \ -0.1\ \ ^3 = -0.000441943598486$ so we must calculate $\xi = \ln\ 6 \cdot 0.441943598486\ \ = 0.9751864589460811$ and we note that this value lies between 0.9 and 1.

8. Use some form of mathematical software to plot

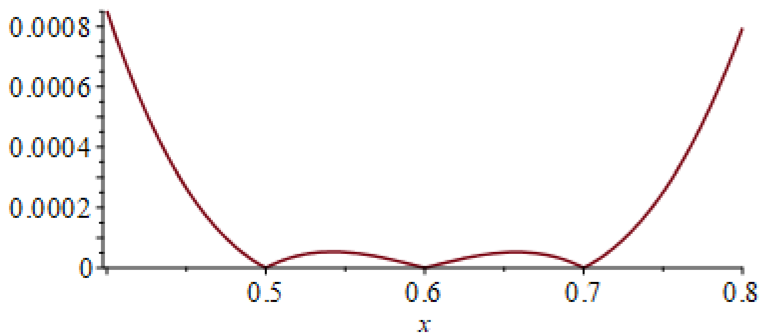
$$p(x) = -0.2820860474088350x^2 + 1.162464000058042x - 0.0312849495726095$$

and sin(*x*) on the interval [0.4, 0.8]. How would you describe the quadratic polynomial, and how does the error $|p(x) - \sin(x)|$ contrast between [0.5, 0.7] and outside this interval?

While the functions appear to be similar on the interval [0.4, 0.8], as shown here,



the absolute error on the interval [0.5, 0.7] does not exceed 0.000054, but outside that interval, the error grows very quickly by over an order of magnitude over a very short distance.



9. Suppose you have an array containing 101 entries, you know the entries are sorted, and you understand that they are approximately uniformly distributed. You know that

`array[0] == 5.3` and `array[100] == -8.3`

You are trying to determine if one of the array entries is equal to 0.0. Which entry of the array would you search next if you were using a binary search, and if you were using an interpolating search?

For a binary search, we would check `array[50]`, but for an interpolation search, we would find the root of the interpolating linear polynomial connecting (0, 5.3) and (100, –8.3), which is 38.97058823529412, and this suggests we should next check `array[39]`.

10. Use your calculator to find a point on the interval [10, 15] that equals the average of the values sin(10), sin(11), sin(12), sin(13), sin(14) and sin(15).

$$\frac{\sin 10 + \sin 11 + \sin 12 + \sin 13 + \sin 14 + \sin 15}{6} = -0.003253667126979967$$

but $\sin^{-1} -0.003253667126979967 = -0.003253672867750357$, which lies outside the interval [10, 15]. The sine function is, however, multivalued, so $\sin(x + 2n\pi) = \sin(x)$ and thus we note that therefore it follows that $x = -0.003253672867750357 + 4\pi = 12.56311694149142$ is the value we're looking for, and if you calculate sin(12.56311694149142), you will see that it equals the average value calculated above.

11. Does a function have to be just continuous or also differentiable for the intermediate-value theorem to apply?

It is only necessary that the function is continuous for the IVT to apply. (For example, although beyond the scope of this course, the Weierstrass function is continuous everywhere and differentiable nowhere, and yet, the graph shows that there must be a root between 0 and 1, as $f(0) = 2$ and $f(1) = -2$.)

12. We have the rule that when rounding, if the digit immediately after the digit to which we are rounding is a 5 and all subsequent digits are zero, half the time we round up, and the other half we round down (based on the digit we are rounding to being odd or even, respectively). If we chose to either only round up or only round down in this case, what type of error would this introduce into our calculations?

If each time we rounded up, then over thousands and millions of calculations, this would slightly bias our answers towards begin greater than the actual result. This would be a systematic error, as this error is added with each calculation that results a calculated result that is half way between the two closest fixed-precision representations.

13. In this course, what are the primary sources of errors we are interested in minimizing?

Numeric error resulting from the use of fixed-precision floating-point numbers affecting both precision and accuracy, and systematic errors that result in approximations that are less accurate. When interpreting data from the real world, we are also dealing with noise or random errors.

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